International Astronomy and Astrophysics Competition
Pre-Final Round 2020

Solutions to the Pre-Final Round 2020

Please note that there are many ways to reach the final solutions.
Not all detailed steps are elaborated in this solution document.

Problem A.1: Interstellar Mission (4 Points)

You are on an interstellar mission from the Earth to the 8.7 light-years distant star Sirius. Your spaceship can travel with 70% the speed of light and has a cylindrical shape with a diameter of 6 m at the front surface and a length of 25 m. You have to cross the interstellar medium with an approximated density of 1 hydrogen atom/m^3.

(a) Calculate the time it takes your spaceship to reach Sirius.
(b) Determine the mass of interstellar gas that collides with your spaceship during the mission.

Note: Use 1.673 $\times$ 10^{-27} kg as proton mass.

Solution a:

$$t = \frac{8.7\,ly}{0.7c} = \frac{8.7\,c \cdot yr}{0.7c} = 12.4\,yr$$

Solution b:
Number of collisions with atoms:

$$N = \rho \cdot V = \rho \cdot A \cdot s = \rho \cdot \pi \left(\frac{d}{2}\right)^2 \cdot s$$

Total mass of interstellar gas:

$$M = m_{H_2} \cdot N = 2m_p \cdot N = 2m_p \cdot \rho \cdot \pi \left(\frac{d}{2}\right)^2 \cdot s = 7.8 \cdot 10^{-9}\,kg$$

Problem A.2: Time Dilation (4 Points)

Because you are moving with an enormous speed, your mission from the previous problem A.1 will be influenced by the effects of time dilation described by special relativity: Your spaceship launches in June 2020 and returns back to Earth directly after arriving at Sirius.

(a) How many years will have passed from your perspective?
(b) At which Earth date (year and month) will you arrive back to Earth?

Solution a:
$\rightarrow$ No time dilation: 24.8 years

Solution b:
With time dilation:

$$t_{Earth} = \frac{t_{Spaceship}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t_{Spaceship}}{\sqrt{1 - 0.7^2}} = 34.7\, yr$$

$\rightarrow$ Date of arrival: January 2055

Problem A.3: Magnitude of Stars (4 Points)

The star Sirius has an apparent magnitude of -1.46 and appears 95-times brighter compared to the more distant star Tau Ceti, which has an absolute magnitude of 5.69.

(a) Explain the terms *apparent magnitude*, *absolute magnitude* and *bolometric magnitude*.
(b) Calculate the apparent magnitude of the star Tau Ceti.
(c) Find the distance between the Earth and Tau Ceti.

Solution a:
$\rightarrow$ Apparent magnitude: brightness observed from Earth (relative scale, historical background)
$\rightarrow$ Absolute magnitude: apparent magnitude in 10 parsecs distance
$\rightarrow$ Bolometric magnitude: including all wavelengths (not only visible light)

Solution b:
The apparent magnitudes are denoted by m_S, m_τ and it is $I_S/I_\tau = 95$:

$$m_\tau = m_S + 2.5 \cdot \log(I_S/I_\tau) = 3.49$$

Solution c:
The absolute magnitude is denoted by M:

$$m - M = 5\log(r) - 5 \implies r = 10^{\frac{m-M+5}{5}} = 3.63\,pc$$

Problem A.4: Emergency Landing (4 Points)

Because your spaceship has an engine failure, you crash-land with an emergency capsule at the equator of a nearby planet. The planet is very small and the surface is a desert with some stones and small rocks laying around. You need water to survive. However, water is only available at the poles of the planet. You find the following items in your emergency capsule:

- Stopwatch
- Electronic scale
- 2m yardstick
- 1 Litre oil
- Measuring cup

Describe an experiment to determine your distance to the poles by using the available items.

Hint: *As the planet is very small, you can assume the same density everywhere.*

Solution:

1. You collect a small rock from the surface.
2. You measure the mass m' of the rock (on this planet) with the electronic scale. The Earth mass m can be determined with the acceleration g (see 5): $m = 1N \cdot m'/g$
3. By using the measuring cup and the oil, you determine the volume V of the rock.
4. This gives you the density of the rock $\rho = m/V$. As the planet is small, you assume this density for the planet. The formula for the mass of the planet is $M = \rho \cdot V = \rho \cdot \frac{4}{3}\pi R^3$.
5. You place the yardstick vertically into the air and drop a small rock from a height h down to the ground. You measure the falling time t with your stopwatch. This let's you determine the acceleration of the stone: $g = 2h/t^2$
6. From Newton's law of universal gravitation it follows that the gravitational acceleration at the surface is $g = G\frac{M}{R^2}$ with the gravitational constant G. By using the formula for the planet's mass, you determine the radius of the planet: $R = \frac{3g}{4\pi G\rho}$
7. You know that you have landed at the equator of the planet. Using basic geometry, you then determine the distance from the equator to the poles: $d = \frac{2\pi R}{4} = \frac{\pi}{2}R$

Problem B.1: Temperature of Earth (6 Points)

Our Sun shines bright with a luminosity of 3.828 x 10^{26} Watt. Her energy is responsible for many processes and the habitable temperatures on the Earth that make our life possible.

(a) Calculate the amount of energy arriving on the Earth in a single day.
(b) To how many litres of heating oil (energy density: 37.3 x 10^{6} J/litre) is this equivalent?
(c) The Earth reflects 30% of this energy: Determine the temperature on Earth's surface.
(d) What other factors should be considered to get an even more precise temperature estimate?

Note: The Earth's radius is 6370 km; the Sun's radius is 696 x 10^{3} km; 1 AU is 1.495 x 10^{8} km.

Solution a:
The energy is distributed on a sphere with radius 1 AU. The surface pointing to the Sun is πR_E^2:

$$E_{day} = L_{\odot} \cdot \frac{A_E}{A_{1AU}} \cdot t = L_{\odot} \cdot \frac{\pi R_E^2}{4\pi \cdot (1AU)^2} \cdot t$$

$\rightarrow$ Result: 1.7 x 10^{17} W, which is 1.5 x 10^{22} J/day

Solution b:
From $V = E_{day}/\rho_E$ it follows 4.0 x 10^{14} Litres.

Solution c:
We use the Stefan-Boltzmann law with $\sigma = 5.67 \cdot 10^{-8} \frac{W}{m^2 K^4}$ and an emissivity ε of 0.7:

$$L_E \cdot \varepsilon = 4\pi R_E^2 \varepsilon \sigma T_E^4 \implies T_E = \sqrt[4]{\frac{L_E \cdot \varepsilon}{4\pi R_E^2 \sigma}}$$

$\rightarrow$ Result: 254.7 K, which is -18.5 °C

Solution d:
$\rightarrow$ the greenhouse effect, layered structure of the atmosphere, etc.

Problem B.2: Distance of the Planets (6 Points)

The table below lists the average distance R to the Sun and orbital period T of the first planets:

	Distance	**Orbital Period**
Mercury	0.39 AU	88 days
Venus	0.72 AU	225 days
Earth	1.00 AU	365 days
Mars	1.52 AU	687 days

(a) Calculate the average distance of Mercury, Venus and Mars to the Earth.
Which one of these planets is the closest to Earth on average?
(b) Calculate the average distance of Mercury, Venus and Earth to Mars.
Which one of these planets is the closest to Mars on average?
(c) What do you expect for the other planets?

Hint: *Assume circular orbits and use symmetries to make the distance calculation easier. You can approximate the average distance by using four well-chosen points on the planet's orbit.*

Solution:
Assuming circular orbits, the position $\vec{r}_i(t)$ of a planet i at a given time t is

$$\vec{r}_i(t) = R_i \begin{pmatrix} \cos(\omega_i t) \\ \sin(\omega_i t) \end{pmatrix}$$

with the angular velocity $\omega = 2\pi/T$. The distance of two planets at the time t is given by

$$\Delta r(t) = |\vec{r}_1(t) - \vec{r}_2(t)| = \sqrt{[R_1 \cos(\omega_1 t) - R_2 \cos(\omega_2 t)]^2 + [R_1 \sin(\omega_1 t) - R_2 \sin(\omega_2 t)]^2}.$$

The circular symmetries allow us to fix a single point of planet 2 for averaging over time:

$$\Delta r(t) = \sqrt{[R_1 \cos(\omega_1 t) - R_2]^2 + [R_1 \sin(\omega_1 t)]^2}$$

Instead of averaging over all t, an approximation with four equally distributed points is sufficient:

$$\langle \Delta r \rangle = \frac{1}{4} \cdot \sum_{k=1}^{4} \Delta r \left(\frac{T_1}{4} k \right)$$

Solution a:

	Average Distance	(without approximation)
Mercury-Earth	1.04 AU	1.04 AU
Venus-Earth	1.12 AU	1.13 AU
Mars-Earth	1.67 AU	1.69 AU

→ Mercury is the closest to Earth.

Solution b:

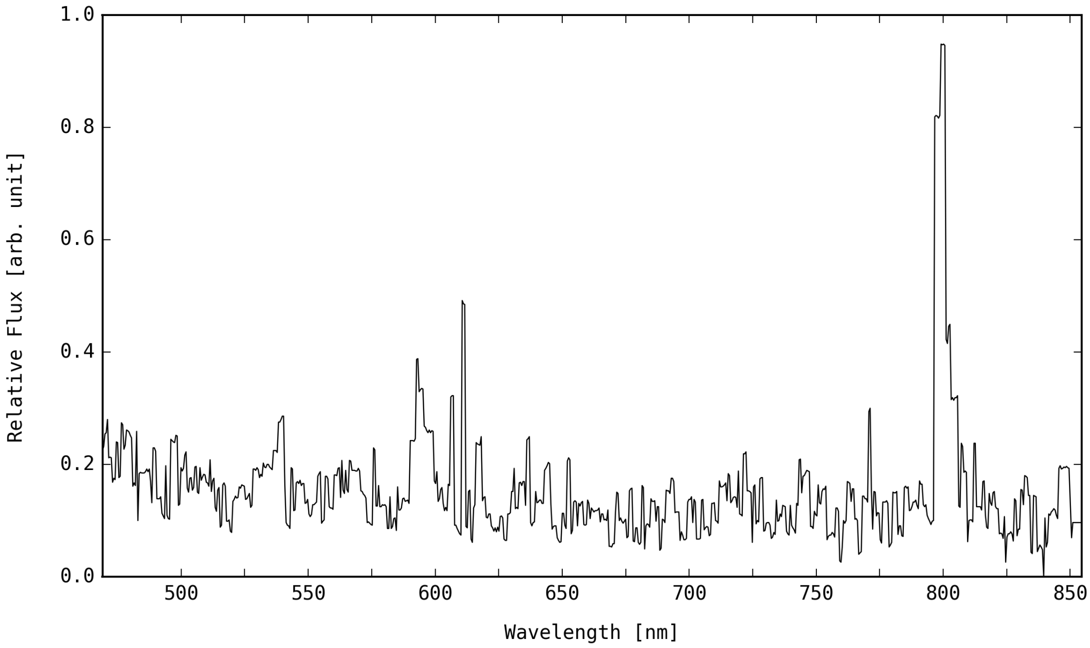
	Average Distance	(without approximation)
Mercury-Mars	1.54 AU	1.55 AU
Venus-Mars	1.60 AU	1.61 AU
Earth-Mars	1.67 AU	1.69 AU

→ Mercury is the closest to Mars.

Solution c:
Mercury is on average the closest planet to all planets of the solar system.

Problem B.3: Mysterious Object (6 Points)

Your research team analysis the light of a mysterious object in space. By using a spectrometer, you can observe the following spectrum of the object. The Hα line peak is clearly visible:



(a) Mark the first four spectral lines of hydrogen (Hα, Hβ, Hγ, Hδ) in the spectrum.
(b) Determine the radial velocity and the direction of the object's movement.
(c) Calculate the distance to the observed object.
(d) What possible type of object is your team observing?

Solution a:
The Hα peak is located at 800 nm (without red shift: 656 nm). This yields the red shift z:

$$z = \frac{\lambda_{obs}}{\lambda_{exp}} - 1 = 0.22$$

From $\lambda_{obs} = (z+1) \cdot \lambda_{exp}$ it follows:

	Hα	Hβ	Hγ	Hδ
At rest	656 nm	486 nm	434 nm	410 nm
With red shift	800 nm	593 nm	529 nm	500 nm

Solution b:
The red shift corresponds to the radial velocity of an object:

$$z = \sqrt{\frac{1+v/c}{1-v/c}} - 1 \;\; \Rightarrow \;\; v = \frac{(z+1)^2 - 1}{(z+1)^2 + 1} c = 58881\ km/s$$

$\rightarrow$ positive red shift, i.e. the object is moving away

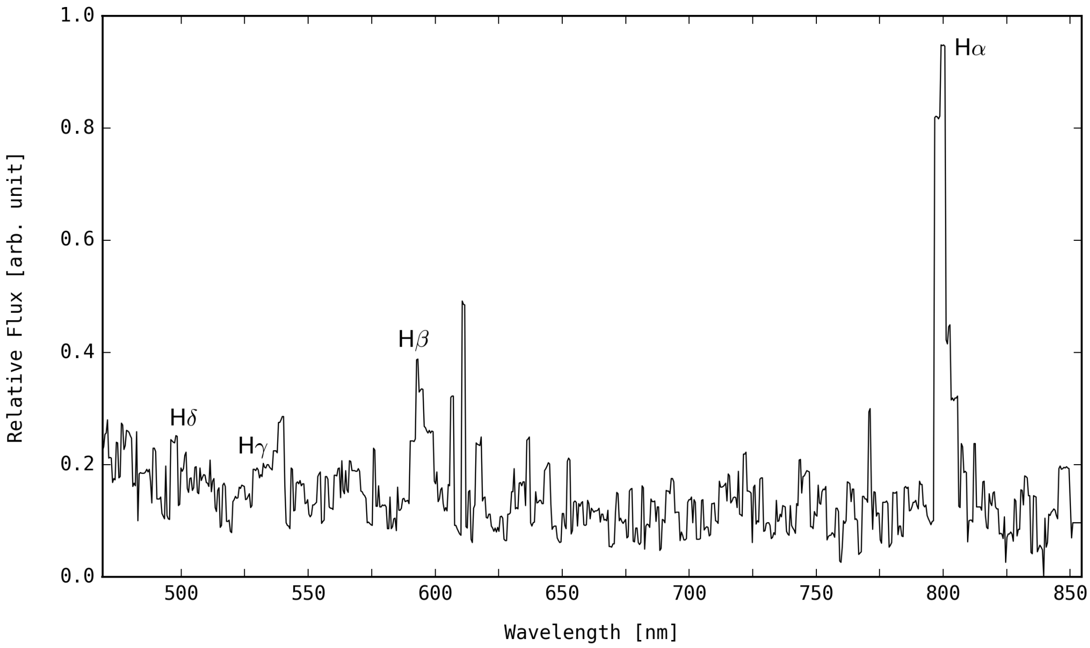
Solution c:

According to Hubble's law, $v = H_0 \cdot d$ with the distance d and the constant H_0 = 70 km/s/Mpc:

$$d = \frac{v}{H_0} = 841\ Mpc$$

Solution d:

$\rightarrow$ possible objects for this large distance: galaxy or quasar



Problem B.4: Distribution of Dark Matter (6 Points)

The most mass of our Milky Way is contained in an inner region close to the core with radius R_0. Because the mass outside this inner region is almost constant, the density distribution can be written as following (assume a flat Milky Way with height z_0):

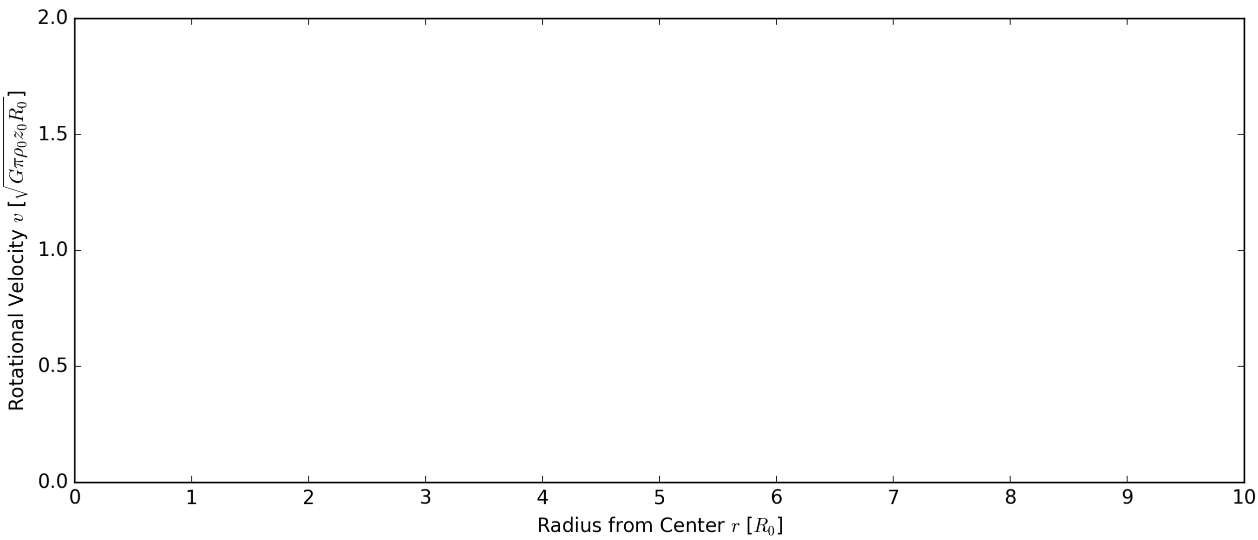
$$\rho(r) = \begin{cases} \rho_0, & r \leq R_0 \\ 0, & r > R_0 \end{cases}$$

(a) Derive an expression for the mass $M(r)$ enclosed within the radius r.
(b) Derive the expected rotational velocity of the Milky Way $v(r)$ at a radius r.

(c) Astronomical observations indicate that the rotational velocity follows a different behaviour:

$$v_{obs}(r) = \sqrt{G\pi\rho_0 z_0 R_0}\left(\frac{5/2}{1+e^{-4r/R_0}} - \frac{5}{4}\right)$$

Draw the expected and observed rotational velocity into the plot below:



(d) Scientists believe the reasons for the difference to be *dark matter*: Determine the rotational velocity due to dark matter $v_{DM}(r)$ from R_0 and draw it into the plot above.
(e) Derive the dark matter mass $M_{DM}(r)$ enclosed in r and explain its distributed.
(f) Explain briefly three theories that provide explanations for *dark matter*.

Solution a:

It is $M(r) = V(r) \cdot \rho_0$ with a volume of $V(r) = \pi r^2 z_0$:

$$M(r) = \begin{cases} \rho_0 \cdot \pi r^2 z_0, & r \leq R_0 \\ \rho_0 \cdot \pi R_0^2 z_0, & r > R_0 \end{cases}$$

Solution b:

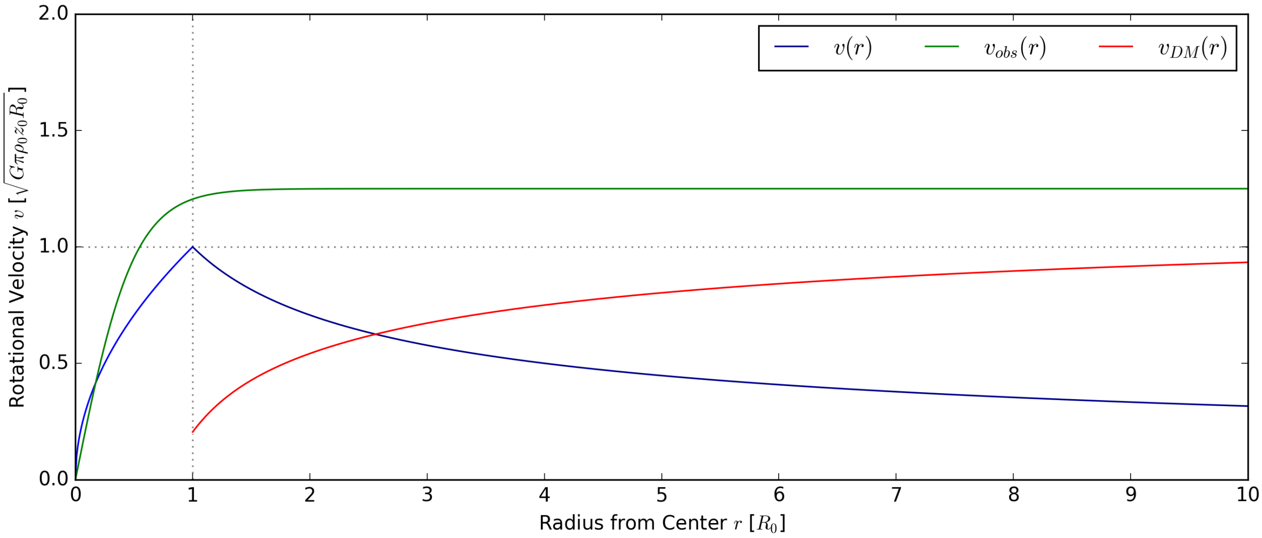
Using basic mechanics we get:

$$a_r = a_g \;\Rightarrow\; \frac{v^2(r)}{r} = G\frac{M(r)}{r^2} \;\Rightarrow\; v(r) = \sqrt{G\frac{M(r)}{r}}$$

Using $M(r)$ we can write the final answer as:

$$v(r) = \begin{cases} \sqrt{G\pi\rho_0 z_0} \cdot \sqrt{r}, & r \leq R_0 \\ \sqrt{G\pi\rho_0 z_0} \cdot R_0/\sqrt{r}, & r > R_0 \end{cases}$$

Solution c:



Solution d:

The difference between observed and expected rotational velocity:

$$v_{DM}(r) = v_{obs}(r) - v(r) = \sqrt{G\pi\rho_0 z_0 R_0}\left(\frac{5/2}{1+e^{-4r/R_0}} - \frac{5}{4} - \sqrt{\frac{R_0}{r}}\right)$$

Solution e:

$\rightarrow$ It is $M_{DM}(r) = v_{DM}^2(r) r/G$; dark matter seems to increase with increasing r (halo).

Solution f:

$\rightarrow$ Examples: WIMPS, Axions, Sterile Neutrinos, Black Holes, MACHOS, MOND, etc.

Problem C.1 : Detection of Gravitational Waves (10 Points)

This problem requires you to read the following recently published scientific article:

Observation of Gravitational Waves from a Binary Black Hole Merger.
B. P. Abbott et al., LIGO Scientific Collaboration and Virgo Collaboration
arXiv:1602.03837, (2016). Link: https://arxiv.org/pdf/1602.03837.pdf

Answer following questions related to this article:

(a) How was the existence of gravitational waves first shown?

→ by the discovery of the binary pulsar system PSR B1913+16 and observations of its energy loss

(b) Which detectors exist around the world? Why did only LIGO detect GW150914?

→ TAMA 300 (Japan), GEO 600 (Germany), LIGO (United States), Virgo (Italy)
→ only LIGO detectors were observing at the time of GW150914

(c) Explain the components of the LIGO detectors.

→ two detectors in Hanford and Livingston (10ms light distance)
→ Michelson interferometer (1064-nm Nd:YAG laser, 4km arm length)
→ resonant optical cavity (multiplies the effect of a gravitational wave on the light)
→ transmissive power-recycling mirror at the input (additional resonant buildup of the laser)
→ transmissive signal-recycling mirror at the output (broadening of bandwidth)

(d) Describe the different sources of noise. How was their impact reduced?

→ seismic noise: quadruple-pendulum system, active seismic isolation platform
→ thermal noise: low-mechanical-loss materials
→ optical phase fluctuations: very low pressure in tubes
→ environmental disturbances: seismometers, accelerometers, microphones, magnetometers, radio receivers, weather sensors, ac-power line monitors, cosmic-ray detector

(e) What indicates that the gravitational wave originated from the merger of a black hole?

→ the objects must be very close and very compact; neutron star pair: insufficient mass; black hole and neutron star pair: merge at much lower frequency
→ decay of the waveform consistent with the oscillations of a relaxing black hole

$\rightarrow$ consistency checks: mass/spin of final black hole, waveform power series, graviton properties

(f) Which are the methods used to search for gravitational wave signals in the detector data?

$\rightarrow$ generic transient search: minimal assumptions about waveforms
$\rightarrow$ binary coalescence search: using waveforms predicted by general relativity

(g) How were the source parameters (mass, distance, etc.) determined from the data?
$\rightarrow$ estimate: with matched-filter search; refinement: general relativity-based models

Problem C.2 : First Image of a Black Hole (10 Points)

This problem requires you to read the following recently published scientific article:

First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole.
The Event Horizon Telescope Collaboration, arXiv:1906.11238, (2019). Link: https://arxiv.org/pdf/1906.11238.pdf

Answer following questions related to this article:

(a) Calculate the photon capture radius and the Schwarzschild radius of M87* (in AU).

$\rightarrow$ photon capture radius: $R_c = \sqrt{27} r_g = \sqrt{27} GM/c^2 = 333\ AU$
$\rightarrow$ Schwarzschild radius: $R_S = 2r_g = 2GM/c^2 = 128\ AU$

(b) Why was it not possible for previous telescopes to take such a picture of the black hole?

$\rightarrow$ due to limited baseline coverage

(c) Describe the components and functionality of the event horizon telescope.

$\rightarrow$ very-long-baseline interferometry; measures visibility of radio brightness directly
$\rightarrow$ observation at 1.3 mm with eight stations over six geographical locations
$\rightarrow$ the separate telescopes simultaneously sample and record the radiation field from the source
$\rightarrow$ synchronization with GPS; equipped with hydrogen maser frequency standard

(d) Explain the two algorithms used to reconstruct the image from the telescope data.

$\rightarrow$ CLEAN: inverse-modeling approach, deconvolves the interferometer point-spread function from the Fourier-transformed visibilities
$\rightarrow$ RML (regularized maximum likelihood): forward-modeling approach, searches for an image that is consistent with the observed data and favours specified image properties

(e) What parameters were required for the GRMHD simulations to generate an image?

$\rightarrow$ properties of the fluid (magnetic field, velocity field, and rest-mass density), the emission and absorption coefficients, the inclination, the position angle, the black hole mass and distance

(f) Explain the physical origins of the features in Figure 3 (central dark region, ring, shadow).

$\rightarrow$ emission ring and shadow: combination of an event horizon and light bending
$\rightarrow$ north-south asymmetry: produced by strong gravitational lensing and relativistic beaming
$\rightarrow$ central flux depression: observational signature of the black hole shadow

(g) How can the image resolution be increased in future observations?

$\rightarrow$ shorter wavelength, by adding more telescopes and space-based interferometry