International Astronomy and
Astrophysics Competition
Pre-Final Round 2021

Solutions to the Pre-Final Round 2021

Please note that there are many ways to reach the final solutions.
Not all detailed steps are elaborated in this solution document.

Problem A.1: Equatorial Coordinate System (4 Points)

Astronomers need to identify the position of objects in the sky with very high precision. For that, it is essential to have coordinate systems that express the position of an object at a given time. One of them is the *equatorial coordinate systems* that is widely used in astronomy.

(a) Explain how the equatorial coordinate system works.
(b) What is the meaning of *J2000* that often occurs together with equatorial coordinates?

The object NGC 4440 is a galaxy located in the Virgo Cluster at the following equatorial coordinates (J2000): $12^h\,27^m\,53.6^s$ (right ascension), $12^\circ\,17'\,36''$ (declination). The Calar Alto Observatory is located in Spain at the geographical coordinates 37.23°N and 2.55°W.

(c) Is the NGC 4440 galaxy observable from the Calar Alto Observatory?

Solution a:
$\rightarrow$ Origin: Earth's center, 0° Latitude: Celestial equator , 0° Longitude: Vernal equinox (intersection between celestial equator and ecliptic), Coordinates: Declination (the angle between object and celestial equator, $\pm 90^\circ$) and right ascension (the angle between vernal equinox and object, 360° or 24h)

Solution b:
$\rightarrow$ Orientation of reference frame is not fixed due to precession, nutation, and other movements; reference point in time for the equinox is necessary, the epoch; J2000 refers to midday on the 1. January 2000

Solution c:
It is $37.23^\circ - 12^\circ\,17'\,36'' < 90^\circ$, thus NGC 4440 is observable.

Problem A.2: Resolution of Telescopes (4 Points)

Telescopes are an essential tool for astronomers to study the universe. You plan to build your own telescope that can resolve the Great Red Spot on the surface of Jupiter at a wavelength of 600 nm. The farthest distance between the Earth and Jupiter is 968×10^6 km and the Great Red Spot has a current diameter of 16,500 km.

(a) Use the Rayleigh criterion to determine the diameter of the lens' aperture of your telescope that is needed to resolve the Great Red Spot on Jupiter.

Impacts have formed many craters on the Moon's surface. You would like to study some of the craters with your new telescope. The distance between Moon and Earth is 384,400 km.

(b) What is the smallest possible size of the craters that your telescope can resolve?

Solution a:
For the required angular resolution θ we get (z: distance to Jupiter, d: diameter of object):

$$\sin\left(\frac{\theta}{2}\right) = \frac{d}{2z} \quad \Longrightarrow \quad \theta_J = 2\arcsin\left(\frac{d}{2z}\right) = 1.705 \times 10^{-5}$$

With the Rayleigh criterion $\theta_J = 1.22\lambda/D$ we get:

$$D = \frac{1.22\lambda}{\theta}$$

$\rightarrow$ Diameter of the lens' aperture: 4.3 cm

Solution b:
For the minimum crater diameter d we get:

$$d = 2z \cdot \sin\left(\frac{\theta}{2}\right)$$

$\rightarrow$ Smallest possible diameter: 6.6 km

Problem A.3: Total Solar Eclipse (4 Points)

A total solar eclipse occurs when the Moon moves between the Earth and the Sun and completely blocks out the Sun. This phenomena is very spectacular and attracts people from all cultures. However, total solar eclipses can also take place on other planets of the Solar System.

Determine for each of the following moons if they can create a total solar eclipse on their planet.

Moon	**Radius**	**Distance to Planet**	**Planet**	**Distance to the Sun**
Phobos	11 km	9376 km	Mars	228×10^6 km
Callisto	2410 km	1.883×10^6 km	Jupiter	779×10^6 km
Titan	2574 km	1.222×10^6 km	Saturn	1433×10^6 km
Oberon	761 km	0.584×10^6 km	Uranus	2875×10^6 km

Note: The radius of the Sun is 696×10^3 km.

Solution:

For a total eclipse, the planet must be within the shadow of the moon (umbra). The distance between the moon and the planet d must be smaller than the distance of the shadow d_U. Geometry gives us (D: distance planet–Sun, R_S: radius of the Sun, R: radius of the moon):

$$\frac{D}{R_S} = \frac{d_U}{R} \implies d_U = D \cdot \frac{R}{R_S}$$

With the condition $d \leq d_U$ we get:

Moon	d_U	
Phobos	3603 km	total eclipse **not possible**
Callisto	2.697×10^6 km	total eclipse **possible**
Titan	5.300×10^6 km	total eclipse **possible**
Oberon	3.143×10^6 km	total eclipse **possible**

Problem A.4: Special Relativity - Part I (6 Points)

Special relativity has become a fundamental theory in the 20th century and is crucial for explaining many astrophysical phenomena. A central aspect of special relativity is the transformation from one reference frame to another. The following Lorentz transformation matrix gives the transformation from a frame at rest to a moving frame with velocity v along the z-axis:

$$\begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

where $\beta = v/c$ with c being the speed of light in a vacuum, and γ is the Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

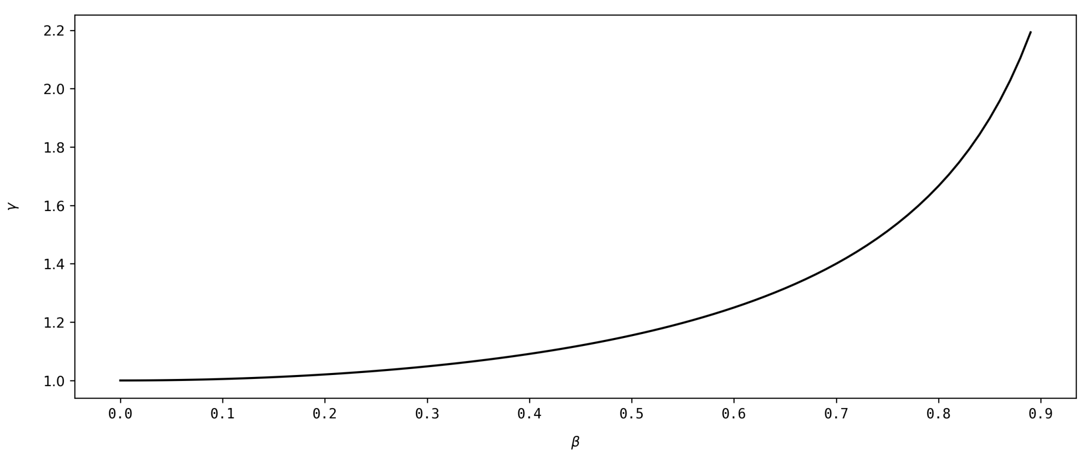
(a) State and explain the two traditional postulates from which special relativity originates.
(b) Draw a plot of the Lorentz factor for $0 \leq \beta \leq 0.9$ to see how its value changes.

One of the many exciting phenomena of special relativity is *time dilation*. Imagine astronauts in a spaceship that is passing by the Earth with a high velocity.

(c) Are the clocks ticking slower for the people on Earth or for the astronauts on the spaceship?
(d) How fast must the spaceship travel such that the clocks go twice as slow?

Solution a:
(1) *The laws of physics are invariant in all inertial frames of reference.*
(2) *The speed of light in vacuum is the same for all observers.*

Solution b:



Solution c:

The clocks are ticking slower for the spaceship.

Solution d:

From the Lorentz factor we get:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad \Longrightarrow \quad \beta = \sqrt{1-\frac{1}{\gamma^2}}$$

With $\gamma = 2$ we have: $\beta = 0.87$, thus 87% of the speed of light

Problem B.1: Space Cannon (4 Points)

Scientists are developing a new *space cannon* to shoot objects from the surface of the Earth directly into a low orbit around the Earth. For testing purpose, a projectile is fired with an initial velocity of 2.8 km/s vertically into the sky.

Calculate the height that the projectile reaches, ...
(a) assuming a constant gravitational deceleration of 9.81 m/s^2.
(b) considering the change of the gravitational force with height.

Note: Neglect the air resistance for this problem. Use 6.67×10^{-11} m^{3}kg^{-1}s^{-2} for the gravitational constant, 6371 km for the Earth's radius, and 5.97×10^{24} kg for the Earth's mass.

Solution a:
Kinetic energy of the projectile:

$$E_{kin} = \frac{1}{2} m v_0^2$$

Potential energy at height h due to the force of gravitation $F_1 = mg$:

$$E_{pot}(h) = \int_0^h F_1 \, dy = mgh$$

The maximum is reached as soon all kinetic energy is converted into potential energy:

$$E_{kin} = E_{pot}(h) \;\Rightarrow\; h = \frac{v_0^2}{2g}$$

$\rightarrow$ Height of the projectile: 400 km

Solution b:
Considering the gravitational force (R Earth's radius, M Earth's mass, G gravitational constant):

$$F_2(y) = G \frac{mM}{(R+y)^2}$$

This yields the potential energy:

$$E_{pot}(h) = \int_0^h F_2(y) \, dy = GmM \int_0^h \frac{1}{(R+y)^2} \, dy = GmM \int_R^{R+h} \frac{1}{r^2} \, dr = GmM \left(\frac{1}{R} - \frac{1}{R+h} \right)$$

With energy conservation it follows:

$$h = \frac{1}{\frac{1}{R} - \frac{v_0^2}{2GM}} - R$$

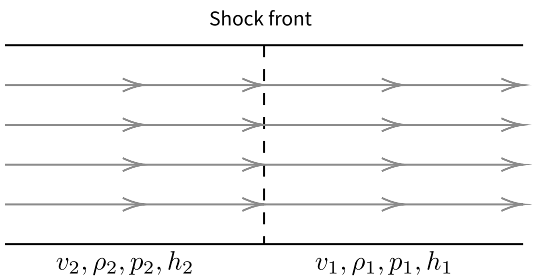
$\rightarrow$ Height of the projectile: 426 km

Problem B.2: Shock Wave (6 Points)

This year's qualification round featured a spaceship escaping from a shock wave (Problem B). The crew survives and wants to study the shock wave in more detail. It can be assumed that the shock wave travels through a stationary flow of an ideal polytropic gas which is adiabatic on both sides of the shock. Properties in front and behind a shock are related through the three Rankine-Hugoniot jump conditions (mass, momentum, energy conservation):

$$\rho_1 v_1 = \rho_2 v_2 \qquad \rho_1 v_1^2 + p_1 = \rho_2 v_2^2 + p_2 \qquad \frac{v_1^2}{2} + h_1 = \frac{v_2^2}{2} + h_2$$

where ρ, v, p, and h are the density, shock velocity, pressure, and specific enthalpy in front ($_1$) and behind ($_2$) the shock respectively.



(a) Explain briefly the following terms used in the text above:
- (i) stationary flow
- (ii) polytropic gas
- (iii) specific enthalpy

(b) Show with the Rankine-Hugoniot conditions that the change in specific enthalpy is given by:

$$\Delta h = \frac{p_2 - p_1}{2} \cdot \left(\frac{1}{\rho_1} + \frac{1}{\rho_2} \right)$$

The general form of Bernoulli's law is fulfilled on both sides of the shock separately:

$$\frac{v^2}{2} + \Phi + h = b$$

where Φ is the gravitational potential and b a constant.

(c) Assuming that the gravitational potential is the same on both sides, determine how the constant b changes at the shock front.

(d) Explain whether Bernoulli's law can be applied across shock fronts.

Solution a:
(i) stationary flow: fluid properties f (e.g. v, ρ, p) do not change with time, i.e. $\partial_t f = 0$
(ii) polytropic gas: it applies that $p \cdot V^n$ is constant for some index n
(iii) specific enthalpy: enthalpy is the sum of a system's internal energy U and $p \cdot V$, thus $H = U + pV$; specific enthalpy is enthalpy per unit of mass, this $h = H/m$

Solution b:
From momentum conservation we obtain:

$$v_1^2 = \frac{\rho_2 v_2^2 + p_2 - p_1}{\rho_1} \qquad v_2^2 = \frac{\rho_1 v_1^2 + p_1 - p_2}{\rho_2}$$

With energy conservation and mass conservation it then follows:

$$\Delta h = h_2 - h_1 = \frac{v_1^2 - v_2^2}{2} = \frac{1}{2}\left(\frac{\rho_2 v_2^2 + p_2 - p_1}{\rho_1} - \frac{\rho_1 v_1^2 + p_1 - p_2}{\rho_2}\right) = \frac{p_2 - p_1}{2} \cdot \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$$

Solution c:
From Bernoulli's law we get:

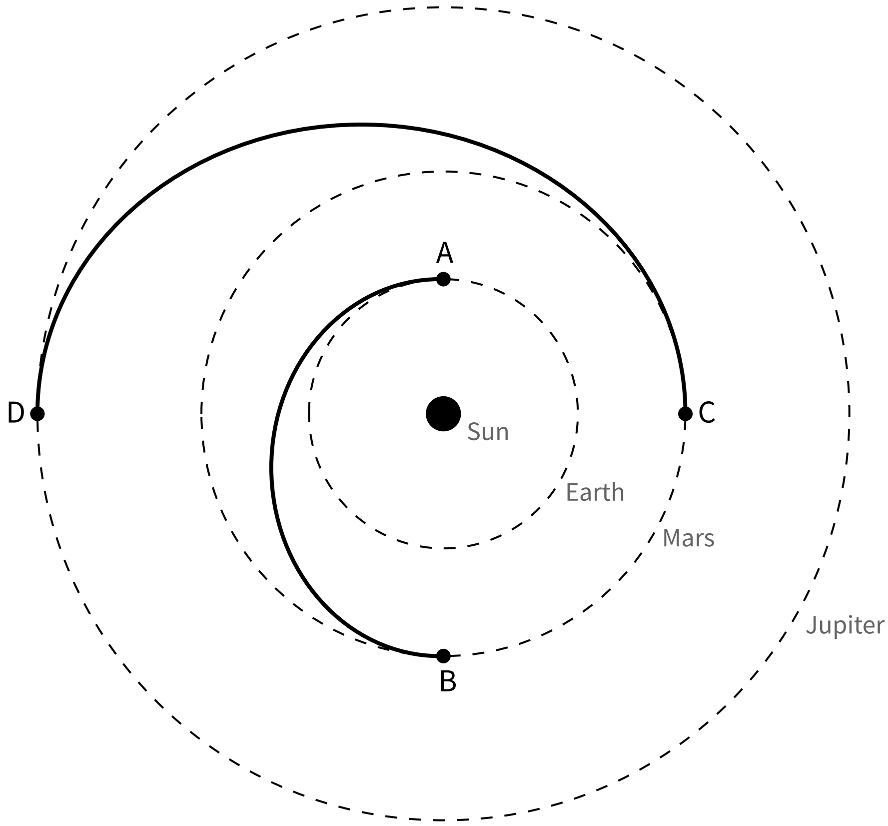
$$b_2 - b_1 = \frac{v_2^2}{2} + \Phi + h_2 - \frac{v_1^2}{2} - \Phi - h_1 = \frac{v_1^2 - v_2^2}{2} + \Delta h = 0$$

Solution c:
As the constant b does not change, Bernoulli's law can be applied across shock fronts.

Problem B.3: Interplanetary Journey (6 Points)

A space probe is about to launch with the objective to explore the planets Mars and Jupiter. To use the lowest amount of energy, the rocket starts from the Earth's orbit (A) and flies in an elliptical orbit to Mars (B), such that the ellipse has its perihelion at Earth's orbit and its aphelion at Mars' orbit. The space probe explores Mars for some time until Mars has completed 1/4 of its orbit (C). After that, the space probe uses the same ellipse to get from Mars (C) to Jupiter (D). There the mission is completed, and the space probe will stay around Jupiter.

The drawing below shows the trajectory of the space probe (not drawn to scale):



Below you find the obrital period and the semi-major axis of the three planets:

	Orbital period	Semi-major axis
Earth	365 days	1.00 AU
Mars	687 days	1.52 AU
Jupiter	4333 days	5.20 AU

How many years after its launch from the Earth (A) will the space probe arrive at Jupiter (D)?

Solution:

This problem requires the use of the third Kepler's laws. The ellipses have a semi-major axis of:

$$R_{12} = \frac{R_1 + R_2}{2}$$

As T^2/R^3 is constant for all planets and ellipses orbiting the Sun, we get (M for Mars):

$$\frac{T_M^2}{R_M^3} = \frac{T_{12}^2}{R_{12}^3} \implies T_{12} = T_M \left(\frac{R_{12}}{R_M}\right)^{3/2} = T_M \left(\frac{R_1 + R_2}{2R_M}\right)^{3/2}$$

The time for the manoeuvre τ is half of the ellipse, thus $\tau_{12} = T_{12}/2$. This gives for A to B:

$$\tau_{AB} = \frac{T_M}{2} \left(\frac{R_E + R_M}{2R_M}\right)^{3/2}$$

which is 518 days. For B to C we have $T_M/4$, thus 172 days. And for C to D:

$$\tau_{CD} = \frac{T_M}{2} \left(\frac{R_M + R_J}{2R_M}\right)^{3/2}$$

which is 1129 days. The total time is then given by:

$$\tau = \tau_{AB} + \frac{T_M}{4} + \tau_{CD}$$

$\rightarrow$ Total time for the mission: 1560 days or 4.2 years

Problem B.4: Special Relativity - Part II (6 Points)

Space and time are interconnected according to special relativity. Because of that, coordinates have four components (three position coordinates x, y, z, one time coordinate t) and can be expressed as a vector with four rows as such:

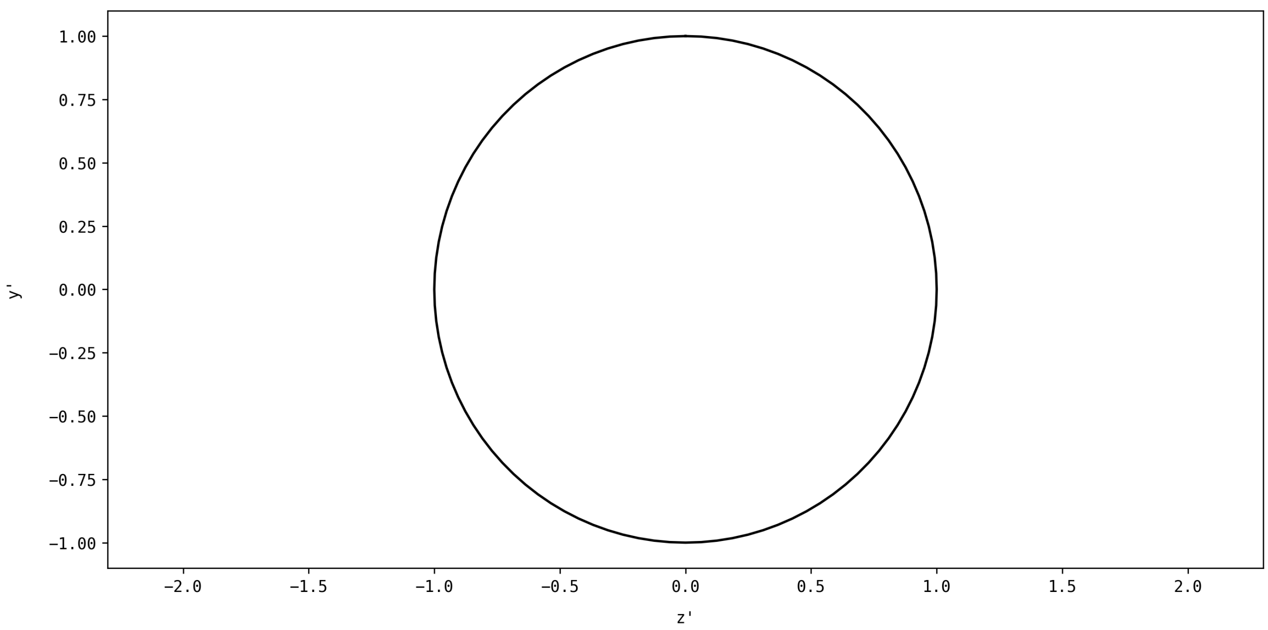
$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

The spaceship from problem A.4 (Special Relativity - Part I) travels away from the Earth into the deep space outside of our Milky Way. The Milky Way has a very circular shape and can be express as all vectors of the following form (for all $0 \leq \varphi < 2\pi$):

$$\begin{pmatrix} ct \\ 0 \\ \sin\varphi \\ \cos\varphi \end{pmatrix}$$

(a) How does the shape of the Milky Way look like for the astronauts in the fast-moving spaceship? To answer this question, apply the Lorentz transformation matrix (see A.4) on the circular shape to get the vectors (ct', x', y', z') of the shape from the perspective of the moving spaceship.

(b) Draw the shape of the Milky Way for a spaceship with a velocity of 20%, 50%, and 90% of the speed of light in the figure below (Note: The ring shape for a resting spaceship is already drawn.):



Solution a:
Applying the Lorentz transformation matrix to get the new shape:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} ct \\ 0 \\ \sin\varphi \\ \cos\varphi \end{pmatrix} = \begin{pmatrix} \gamma ct + \gamma\beta\cos\varphi \\ 0 \\ \sin\varphi \\ \gamma\beta ct + \gamma\cos\varphi \end{pmatrix}$$

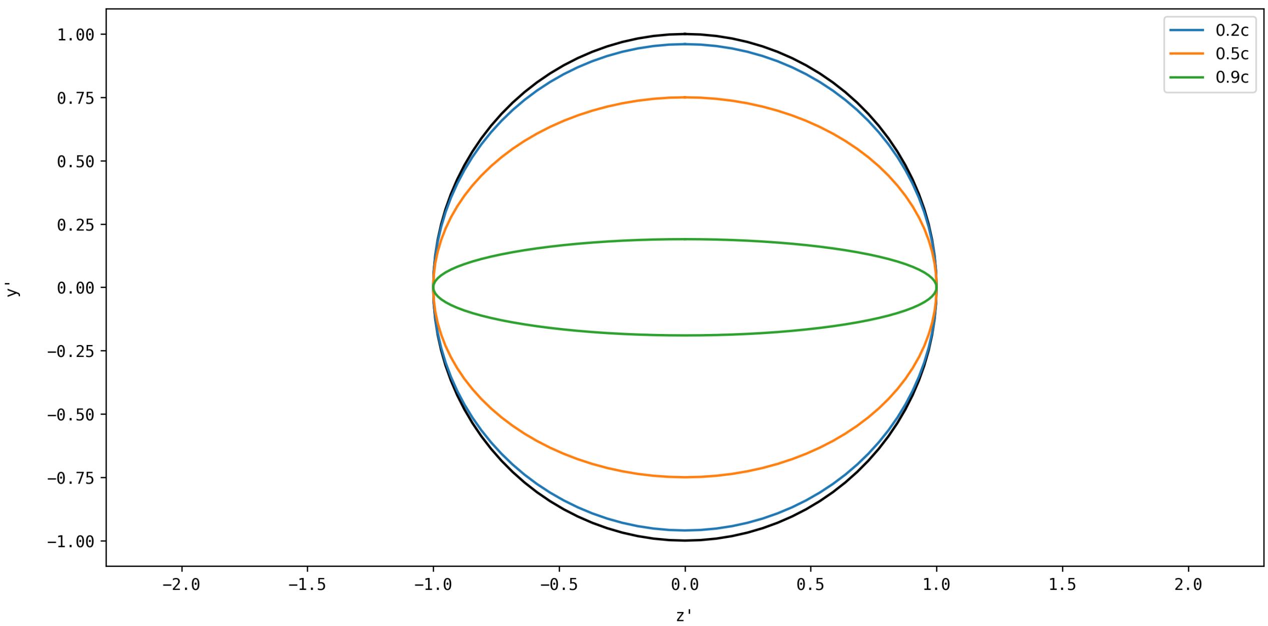
With $ct' = \gamma ct + \gamma\beta\cos\varphi$, we can convert into the time of the moving frame:

$$\begin{pmatrix} ct' \\ 0 \\ \sin\varphi \\ \beta ct' + \gamma(1-\beta^2)\cos\varphi \end{pmatrix}$$

With $\gamma(1-\beta^2) = 1/\gamma$, we then get:

$$\begin{pmatrix} ct' \\ 0 \\ \sin\varphi \\ \beta ct' + \cos\varphi/\gamma \end{pmatrix}$$

Solution b:
We use $t' = 0$ and get due to the length contraction:



Problem C.1 : Earliest Galaxy Group (10 Points)

This problem requires you to read the following recently published scientific article:

Onset of Cosmic Reionization: Evidence of an Ionized Bubble Merely 680 Myr after the Big Bang.
V. Tilvi et al 2020 ApJL 891 L10. Link: https://iopscience.iop.org/article/10.3847/2041-8213/ab75ec/pdf

Answer the following questions related to this article:

(a) What is the so called *cosmic reionization process*?
$\rightarrow$ epoch in universe development: neutral transparent universe from galaxies etc.
$\rightarrow$ start: UV radiation from galaxies/groups ionized local surroundings
$\rightarrow$ end: entire intergalactic medium is ionized

(b) What are Lyα lines and why did the researches aim to observe them?
$\rightarrow$ transition lines of hydrogen, from 2nd to 1st orbit, emission at 121.6 nm (UV)
$\rightarrow$ indication for ionized intergalactic medium

(c) What do the authors intend to point out with Figure 1 (see article)?
$\rightarrow$ significant fluxes in redder wavelengths
$\rightarrow$ no fluxes in visible wavelengths
$\rightarrow$ indication for being at high redshifts

(d) How is confirmed that the peaks seen in Figure 3 are actually from Lyα emissions?
$\rightarrow$ by considering the line asymmetry and calculating the Skewness

(e) How are the bubble sizes of the galaxies estimated?
$\rightarrow$ theoretical model, relation between Lyα luminosity and bubble size predicted via simulations

(f) What is special about the findings and what are the scientific implications?
$\rightarrow$ most distant galaxy group found yet
$\rightarrow$ supports for inhomogeneous reionization through ionized bubbles

Problem C.2 : Massive Protostar Jet (10 Points)

This problem requires you to read the following recently published scientific article:

Measuring the ionisation fraction in a jet from a massive protostar.
Fedriani, R., Caratti o Garatti, A., Purser, S.J.D. et al. Nat Commun 10, 3630 (2019).
Link: https://www.nature.com/articles/s41467-019-11595-x.pdf

Answer the following questions related to this article:

(a) Why are massive stars important for the development of the universe?
$\rightarrow$ synthesising most of the chemical elements
$\rightarrow$ major feedback into the molecular clouds where stars are born

(b) How can the ionised part of jets be observed?
$\rightarrow$ radio continuum emission from protostellar jets interpreted as thermal bremsstrahlung
$\rightarrow$ (radio emission does not suffer from extinction)

(c) What kind of region is G35.2N? Describe how the region is structured.
$\rightarrow$ high-mass star-forming region
$\rightarrow$ two main cores: core A and core B
$\rightarrow$ Core B: binary system, B-type stars

(d) What is the ionisation fraction χ_e and how is it being calculated?
$\rightarrow$ total number density over electron number density: $\chi_e = n_e/n_{tot}$
$\rightarrow$ n_{tot}: from properties of Fe II emission lines
$\rightarrow$ n_e: from ratio of Fe II lines or properties of radio emission

(e) How is the mass-loss rate being determined for knots K3 and K4? Why not for K1 and K2?
$\rightarrow$ mass-loss rate via $\dot{M} = M v_\perp / k_\perp$
$\rightarrow$ mass of knots not directly, but via $M = \mu m_H n_{tot} V$
$\rightarrow$ K1: no velocity information available
$\rightarrow$ K2: only upper limit on n_e given

(f) Why is the ionisation fraction so small for G35.2N?
$\rightarrow$ Explanation 1: stage of high ionisation not yet reached
$\rightarrow$ Explanation 2: still accreting, thus protostar is swollen and too cool to photo-ionise